Stat 135 Lab 6

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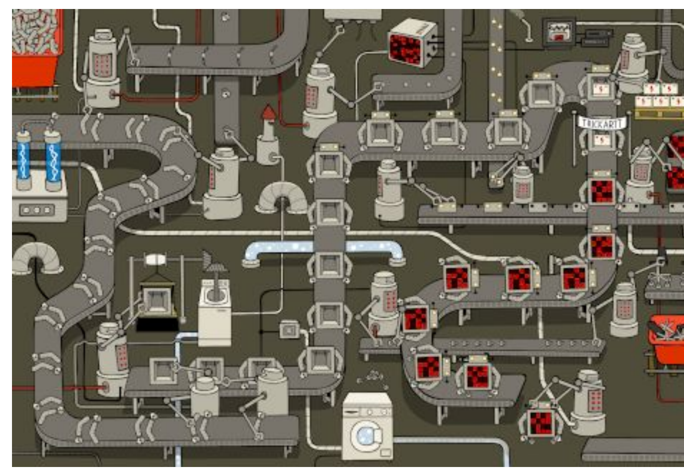
To-do Today:

Notice: I made some schedule changes to make the lab sections more organized.

- In the first 60 min, I will review important concepts and theorems.
- In the second hour, you will be given 30 min to solve all the practice problems, and we will go over them together in the last 30 min.



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To-do Today:

Topics:

1. Sufficiency, minimal sufficiency
1. Hypothesis testing, likelihood ratio test
2. Uniformly most powerful test
3. Generalized likelihood ratio test
4. p-value

Review: Sufficiency

- A **statistic** $T = T(X_1, ..., X_n)$ is a function of the data only. (no parameter θ involved!)
  Examples of a statistic:
  (1) $(X_1, X_2, ..., X_n)$ (2) X_3
  (3) $(X_{(1)}, X_{(2)}, ..., X_{(n)})$ (4) $s^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X})^2$

- A statistic $T = T(X_1, ..., X_n)$ is **sufficient** for $\mathcal{P} = \{P_\theta, \forall\theta \in \Theta\}$ if $P_\theta(X_1, ..., X_n \mid T = t)$ does not depend on θ.

- **(Factorization)** T is sufficient for $\mathcal{P}$ if and only if (iff) there exists functions g_θ, h such that $P_\theta(X^n) = g_\theta(T(X^n))h(X^n)$, where X^n denotes $(X_1, X_2, ..., X_n)$.

- If T is sufficient for θ, the maximum likelihood estimate is a function of T.

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Review: Minimal Sufficiency

- *Definition:* T is **minimal sufficient** for $\mathcal{P}$ if
  (1) T is sufficient,
  (2) for any sufficient $S = S(X^n)$, there exists f with $T = f(S)$

- *Criterion:* T is **minimal sufficient** for $\mathcal{P}$ iff
  $\frac{P_\theta(X^n)}{P_\theta(Y^n)}$ does not depend on $\theta \iff T(X^n) = T(Y^n)$

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Review: Hypothesis Testing

- A **hypothesis** is a statement about the parameter. One hypothesis $H_0 : \theta \in \Theta_0$ is singled out as the **null hypothesis**, and the other complementary one is $H_1 : \theta \in \Theta_1$ as the **alternative hypothesis**.

- Rejecting H_0 when it is true is called a **type I error**.

- The probability of a type I error is called the **significance level** of the test and is usually denoted by α.
  $\alpha = \mathbb{P}_0(d(X) = 1)$

- Accepting the null hypothesis when it is false is called a **type II error** and its probability is usually denoted by β.
  $\beta = \mathbb{P}_1(d(X) = 0)$

- The probability that the null hypothesis is rejected when it is false is called the **power** of the test, and equals $1 - \beta$.
  $1 - \beta = 1 - \mathbb{P}_1(d(X) = 0) = \mathbb{P}_1(d(X) = 1)$

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Review: Likelihood Ratio Test

- A **test statistic** is a function of your data that leads you to a decision whether to reject or not reject the null hypothesis.
- For some fixed significance level α, the **likelihood ratio test** says: reject H_0 if $\Lambda < c$, where $\Lambda = \frac{P_0(X)}{P_1(X)}$ is the likelihood ratio, and c is some function of α, with $\alpha = P_0(d(X) = 1) = P_0(\Lambda < c)$.
- The **rejection region** is the set of values of the test statistic that leads to rejection of H_0.

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Review: Uniformly Most Powerful Test

- A **simple hypothesis** is one that fully specifies the sampling distribution. (Θ_0 or Θ_1 is a singleton.)
  If a hypothesis does not completely specify the probability distribution, the hypothesis is called a **composite hypothesis**.

- **Neyman-Pearson Lemma**: Suppose that H_0 and H_1 are simple hypotheses and that LRT rejects H_0 with significance level α, then any other level-α test has smaller power.

- If the null H_0 is simple and the alternative H_1 is composite, a test that is most powerful for every simple alternative in H_1 is said to be **uniformly most powerful**.

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Review: Generalized likelihood ratio test

- The likelihood ratio test is optimal for testing a simple hypothesis versus a simple hypothesis. And **generalized LRT** is used when the hypotheses are not simple.
- Suppose $H_0 : \theta \in \Theta_0, H_1 : \theta \in \Theta_1$, where $\Theta_0 \cap \Theta_1 = \emptyset, \Omega = \Theta_0 \cup \Theta_1$. Define

$$\Lambda = \frac{\max_{\theta \in \Theta_0}[\text{Lik}(\theta)]}{\max_{\theta \in \Omega}[\text{Lik}(\theta)]}$$

  , where $\max_{\theta \in \Omega}[\text{Lik}(\theta)] = \text{Lik}(\hat{\theta}_{\text{ML}})$.
- GLRT: reject H_0 if $\Lambda < c$, where c is some function of α.

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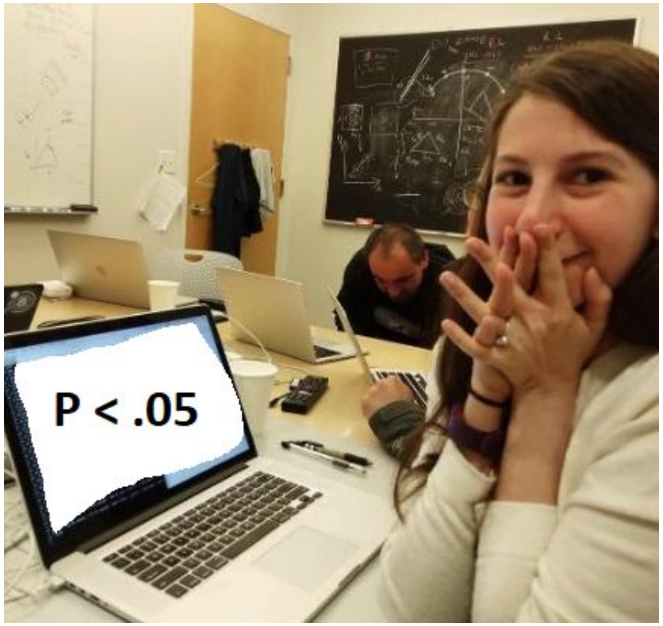
Preview/Review: p-value

- **p-value** is the probability of getting a test statistic as or more extreme as what you observed, given the null hypothsis being true.



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Caveats about p-value!

Nature @nature · Apr 2, 2019

Editorial: Using P values as the sole arbiter of what to accept as truth can mean that some analyses are biased, some false positives are overhyped and some genuine effects are overlooked.

It’s time to talk about ditching statistical significance

Looking beyond a much used and abused measure would make science harder, but better.

nature.com

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Let's Go! 💪

What: to solve lab problems

How long: 30-45 min

Who: by yourself or a study group of 2-3 ppl

Where: bCourses/pages/labs - Lab5 Q2, Lab 6

📝 Please make good use of your time and ask your peers or me for any questions.

🤯 I will get fast-paced on board!

Q1: Sufficiency

Suppose you observe $X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim}$ Bernoulli(p). Recall that if $0 \leqslant p \leqslant 1$, then $0 \leqslant p(1-p) \leqslant 1/4$.

(a) What is the MLE of p?

(b) Is $\sum_{i=1}^{n} X_i$ sufficient for p? Show that X_1 is not sufficient but $X_1, \ldots, X_n$ (i.e., the sample itself) is sufficient.

(c) Find the estimator with minimum variance among all unbiased estimators. Explain why your choice has this property.

(d) Suppose you want to estimate $\theta = p(1-p)$. Find an unbiased estimator $\hat{\theta}$ of θ that depends only on $\overline{X}_n$.

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Q2: Likelihood Ratio Test

Let $X = (X_1, \ldots, X_n)$ be i.i.d $N(\mu, \sigma^2)$ with known σ^2.

$$H_0 : \mu = \mu_0$$

$$H_1 : \mu = \mu_1$$

$$\alpha = 0.01$$

(a). What is the rejection region for this test?
(b). Is $H_0 : \mu = \mu_0$, $H_1 : \mu \neq \mu_0$ a uniformly most powerful test?
(c). Compute the power of the tset for $\mu_0 < \mu_1$ case.

Q3: Likelihood Ratio Test (Rice 9.11.4 a, b)

Let X have one of the following distributions:

X	H_0	H_1
x_1	0.2	0.1
x_2	0.3	0.4
x_3	0.3	0.1
x_4	0.2	0.4

(a) Compare the likelihood ratio Λ, for each possible value X and order the x_i according to Λ.
(b) What is the likelihood ratio test of H_0 versus H_A at level $\alpha = 0.2$? What is the test at level $\alpha = 0.5$?

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Q4: GLRT (Rice 9.11.24 a)

Let X be a binomial random variable with n trials and probability p of success.
What is the generalized likelihood ratio for testing $H_0 : p = 0.5$ versus $H_A : p = 0.5$?

Q5: Power

Let X be a *single* observation from the probability density function $f(x) = \theta x^{\theta-1}, 0 < x < 1$.

(a) Find the most powerful test using significance level $\alpha = 0.05$ for testing the hypothesis $H_0 : \theta = 1$ and $H_1 : \theta = 2$ (sketch the densities $f(x \mid H_0)$ and $f(x \mid H_1)$ for the two hypothesis).

(b) What is the power of the test?

(c) What is the p-value of $X = 0.8$?

(d) For fixed $\alpha = 0.05$, is the test uniformly most powerful against the alternative hypothesis $H_1 : \theta > 1$?